Chapter 9

9. DIFFERENTIAL EQUATIONS: Introduction

Abstract

In this Chapter we introduce ordinary differential equations (ODEs) as a tool to model dynamics. We present examples of how to formulate them based on the dynamical system that needs to be modeled, and discuss the mathematical techniques one can employ to solve the equation analytically. We show how to solve linear differential equations with and without a forcing term, inhomogeneous and homogeneous ODEs respectively. To illustrate the analysis of these equations, ODEs with first order derivatives (e.g. dc/dt) and second order derivatives (e.g. d^2c/dt^2) are used in the examples. Next, we show how higher order ODEs can be represented as a set of first order ones, and how this leads to a formalism in matrix/vector notation that can be efficiently analyzed using techniques from linear algebra. To complete the overview of the toolkit for solving ODEs, the final part of this Chapter briefly refers to application of Laplace and Fourier transforms to solve them.

Keywords

Dynamics
Ordinary Differential Equation (ODE)
linear homogeneous equation
linear inhomogeneous equation
forcing term
characteristic equation
eigenvalue

9. 1 Modeling Dynamics

When modeling some aspect of a neural system, we can represent static variables with **algebraic expressions**, e.g. the concentration c of some chemical in the brain is ten units, $c = 10$. Of course, we can make these expressions a bit more complicated; for instance the concentration could also depend on some combination of other substances a_1 and a_2: e.g. $c = 5a_1 + a_2 + 10$. It is important to realize that the variables do not change with time or space. For that reason the value of this type of equation is limited to situations where we study properties that remain constant over the range and duration of our interest. In contrast, when we model the **dynamics** of a concentration of a brain substance over time or space, we have to include the rate of change of the variable(s) in the equation. For example, if the speed of change of concentration c is known to be five units per unit time, we could model this with: $v = 5$ and $c = v \times t$, where v – speed of change and t – time. So if we assume an initial condition $c = 1$ at $t = 0$, we find that at time $t = 12$ the concentration is $c = 1 + 5 \times 12 = 61$. We can recast this scenario in a different notation. We start by setting the change of variable c to an expression for velocity: $v = \Delta c/\Delta t$, where Δc is the change in concentration c, and Δt is the time interval required for this change. For example, let us measure c at two instances in time $t1$ and $t2$. Let us say that, at these times, we measured concentrations $c1$ and $c2$ respectively. Now we can compute the average speed v of change in concentration as $v = (c2 - c1)/(t2 - t1)$; here we substituted $\Delta c = c2 - c1$ and $\Delta t = t2 - t1$. If we make the intervals over which we determine change infinitesimally small, and we change the notation for change of concentration and time $\Delta \rightarrow d$, we will obtain the expression for the instantaneous speed, e.g. $v = dc/dt = 5$. The expression

$$\dot{c} = dc/dt = 5 \qquad (9.1)$$

is a **first order ordinary differential equation** (1st order ODE). It's first order because the derivative, using the notation $\dot{c}$ or dc/dt is a first order derivative and it's ordinary since there are no **partial differentials** such as $\partial c/\partial t$. Thus a differential equation is an expression that includes the rate of change of a variable rather than only it's state – this enables us to quantify the dynamics (e.g. change over time) of a variable in a quantitative fashion.

Of course if a state can change over time to quantify the velocity (rate of state change), the velocity itself can also change over time. For example if you drive a car, your speed isn't constant because you can accelerate or decelerate. Similarly, the rate of change of some substances in the brain can also change over time: the speed of built up can accelerate or decelerate. The rate of change of a state was given by $v = dc/dt$. Similarly, we can use dv/dt for the rate of change in speed, i.e. for acceleration a. Now, combining $v = dc/dt$ and $a = dv/dt$, we find that we can also get the rate of change in the speed by taking the derivative from the state c twice! The notation for taking the derivative twice, i.e. a second order derivative, is $a = d^2c/dt^2$. When such a second order term is included in a differential equation, we have a 2nd order ODE. Suppose we find some equation $5 \times c + 6 \times v + 3 \times a = 0$, relating state of substance c, speed of change of c given by v, and rate of change of the speed as acceleration a, then we can rewrite this equation in the calculus notation as a 2nd order ODE:

$$5c + 6\,dc/dt + 3\,d^2c/dt^2 = 0. \qquad (9.2)$$

When written in this form it can be seen that this expression is a second order ordinary differential equation (2nd order ODE) because the highest derivative in the expression d^2c/dt^2 is a second order one, and it's an ordinary differential equation because the expression does not include any partial derivative.

The take-home message here is that an ODE is simply an equation that quantifies dynamics, i.e. the rate of change of some variable (e.g. with respect to time or space) rather than just its state. Although the following introduction in differential equations isn't an extensive treatment of the topic, we will review a set of frequently used cases and their solutions. The material in this Chapter can be considered an introduction to the applications described in Chapters on models and filters. We will discuss ODEs by using examples, and we won't worry too much about 'technical details' such as existence and uniqueness of solutions, but rather deal with these aspects as they occur. Readers familiar with ODEs can skip this Chapter and the next, and for more details I refer to Boas (1966), Jordan and Smith (1997), and Polking et al. (2006).

9.2 How to Formulate an ODE?

The example above shows that differential equations may also include a term with the state of the variable itself, in this case $5c$. An intuitive example of an ODE that includes the state of the variable is the expression for the growth of a bacterial culture. Since each bacterium can divide to produce offspring, the rate of growth g in a bacterial population is proportional to its population size S, i.e. $g = kS$, here k is some value that reflects that proportionality. Since the growth g is the rate of change of S, i.e. dS/dt, we may also write

$$dS/dt = kS. \qquad (9.3)$$

This expression is a 1st order ODE, since the highest order of derivation is 1st order, i.e. dS/dt. Why is it advantageous the write $dS/dt = kS$ and not $g = kS$ (which seems much simpler)? The reason is that the former notation allows one to use the methods of calculus developed for solving differential equations.

Another example of the formulation of an ODE can be illustrated with a (bio)chemical reaction, e.g. a reversible reaction in which substances A and B produce substance C and vice versa:

INSET 1 <u>EXACTLY</u> HERE

Here, the symbols k_1 and k_2 represent the constants that govern the rate of the reactions, and A, B and C denote the concentrations of these substances. Now suppose we are interested in finding the expression for the dynamics for compound C. In the reaction above we observe that production of C is proportional with A and B, and depends on rate constant k_1. In the reversed reaction we see that C's disappearance is proportional with itself and occurs at a rate governed by constant k_2. This leads to an expression for the rate of change for C that includes two terms, i.e. one for the production and one for the disappearance of C:

$$dC/dt = k_1AB - k_2C \qquad (9.4)$$

This example shows how to develop the ODE from the chemical interaction. Although we won't go into the details of this example here, this formalism can then be used to analyze the dynamics of the reaction. One side comment, it can easily be seen in Equation (9.4) that the equilibrium state, $dC/dt = 0$ is determined by the rate constants and occurs at $k_1/k_2 = C/AB$.

9.3 Solving 1st and 2nd Order Ordinary Differential Equations

Examples of first and second order ODEs were given in the introduction above. Here we will discuss how to solve this type of equation. In this Section we will discuss the type of ODE that is called **linear homogeneous equation with constant coefficients**. Alternatively, one can use the term **unforced** instead of homogeneous. These ODEs have the form:

$\dot{S} + cS = 0$ for the 1st order case, or (9.5)
$\ddot{S} + b\dot{S} + cS = 0$ for the 2nd order case,
with b, c - constants

Here we used $\dot{S}$ and $\ddot{S}$ to denote first and second order time derivatives dS/dt and d^2S/dt^2 respectively.

As a start, let us go back to the example of bacterial growth in Equation (9.3). Note that this ODE has indeed the form of a 1st order case in Equation (9.5). This type of expression is relatively straightforward since we can **separate the variables** S and t. Using this property, the expression $dS/dt = kS$ can be rewritten as $dS/S = kdt$. The latter equation consists of two expressions left and right of the equal sign that each can be integrated:

$$\int dS/S = \int kdt. \qquad (9.6)$$

Here it is fairly obvious that a limited range for S applies. Since we have S in the denominator, the equation is only valid for nonzero values of S, a reasonable condition since a bacterial population cannot grow if there are no bacteria. In addition, since the number of bacteria cannot be negative, we must have $S > 0$. Hence we can write the solution of the integrals in Equation (9.6):

$$\ln(S) = kt + C \qquad (9.7)$$

Taking the exponential at both sides of Equation (9.7) gives us:

$$S(t) = e^{kt+C} = e^C e^{kt} = Ae^{kt}. \qquad (9.8)$$

Here we explicitly wrote S as a function of t and replaced the constant e^C by A. We are almost done; the only task remaining is to determine constant A, and we here we will use the so called **initial condition** to find the value of A. Say that at time set to zero ($t = 0$) we found the value for S to be S_0. We can substitute this in equation (9.8) to find the value for the constant A. Recall that for $t = 0$ we have $e^{k0} = 1$, thus using the initial condition, we find: $S(0) = Ae^{k0} = A = S_0$. Now we plug this into Equation (9.8) and obtain the solution to our first order ODE:

$$S(t) = S_0 e^{kt}. \qquad (9.9)$$

Instead of the initial condition, we could have used the value of S at any other point in time to determine A. Using the initial condition is just convenient because at $t = 0$ we have unity for the exponential in the equation, i.e. $e^{k0} = 1$. For this reason, one might simply set $t = 0$ at the point in time that we have measured the value of S.

As we will see in Chapter 11, so-called two dimensional dynamical systems satisfy a 2nd order ODE. Here, the term dimension should not necessarily be taken too literal. For example, the neuronal network activity equations formulated by Wilson and Cowan (1972), described in Chapter 31 is an example of a nonlinear version of such a two dimensional system: in their model, the <u>two</u> dimensions are the activities of <u>two</u> neuronal populations, i.e. an excitatory and an inhibitory population. Here we will start with a general example of a linear 2nd order ODE:

$$a\ddot{S} + b\dot{S} + cS = 0.$$

Note that, without loss of generality, we can simplify Equation (9.8) by dividing by a, i.e. making the coefficient of $\ddot{S}$ equal to 1. This is what we will do in the following, and we will simply write the generic expression for a 2nd order ODE as

$$\ddot{S} + b\dot{S} + cS = 0. \qquad (9.10)$$

Note that this is according to the general expression for the 2nd order case in Equation (9.5). Now let us assume there is a solution in the form of $S = e^{kt}$, a solution of the same form as the one we found for the 1st order ODE. If we compute the derivatives $\dot{S}$ and $\ddot{S}$ of the assumed solution e^{kt}, we get ke^{kt} and k^2e^{kt}. Now plug these in the 2nd order ODE in Equation (9.10):

$$k^2e^{kt} + bke^{kt} + ce^{kt} = 0.$$

This can be simplified to:

$$(k^2 + bk + c)e^{kt} = 0,$$

and since e^{kt} is not generally zero, we can satisfy this equation by the standard quadratic equation

$$k^2 + bk + c = 0, \qquad (9.11)$$

which is called the **characteristic equation** of the ODE. The solution of this equation is:

$$k_{1,2} = \left(-b \pm \sqrt{b^2 - 4c}\right)/2. \qquad (9.12)$$

Here we have, depending on the value of the term under the square root, three possible scenarios:

(1) two real solutions for k if $(b^2 - 4c) > 0$,
(2) two complex solutions $(b^2 - 4c) < 0$, or

(3) just one real solution $(b^2 - 4c) = 0$.

The first two cases (1) and (2) are fairly straightforward if we realize that any scaled superposition of solutions of the equation is a solution of a linear system (see also Chapter 13). Since we found two solutions $e^{k_1 t}$ and $e^{k_2 t}$, we find the general solution here as the sum of the two scaled solutions:

$$S = Ae^{k_1 t} + Be^{k_2 t}, \qquad (9.13a)$$

with A and B scaling constants to be determined, often from known conditions at $t = 0$ and $t = \infty$ (see Homework problem 2). Note that in condition (2), if the solutions for k_1 and k_2 are complex, the expression in Eq. (9.13a) represents an oscillation (this is straightforward to see if you recall Euler's formula: $e^{j\omega t} = \cos \omega t + j \sin \omega t$).

The third case $k_1 = k_2$ only provides one solution which, if applied the same approach as above, would lead to $S = Ae^{k_1 t} + Be^{k_1 t} = Ce^{k_1 t}$, which is clearly not sufficient to describe two dimensional dynamics. This dilemma is resolved by changing the constant into a function of time:

$$S = C(t)e^{k_1 t} \qquad (9.13b)$$

In models of neuronal networks with explicit synaptic function, this type of expression is often referred to as the alpha function and used to model the postsynaptic current (***CH 31***).

One important take-home message when comparing the solutions of the 1st and 2nd order ODEs is that they each have a range of potential behaviors. The first order ODE has a simple exponential $S_0 e^{kt}$ as its solution, and depending on the value of rate constant k, the value of the variable S can increase decrease or remain the same. The 2nd order ODE's solution allows for the same behavior, but there is additional behavior possible, namely oscillation. Although this directly follows from the solutions from these ODEs, it can be seen directly in the so-called phase space representation that we will introduce in Chapter 11.

9.4 Ordinary Differential Equations with a Forcing Term

The ODEs in the previous Section contain a number of terms, potentially including the dynamic variable and its derivative(s). Here we will extend this type of example to a case where there is an additional term that does not include the dynamic variable or its derivative. This results in an equation of the form:

$\dot{S} + cS = f(t)$ for the 1st order case, or (9.14)
$\ddot{S} + b\dot{S} + cS = f(t)$ for the 2nd order case,
with b, c – constants, and $f(t)$ - forcing term

In physical systems the additional term $f(t)$ often represents an external force and therefore it is called the **forcing term**. Due to this term, in contrast to the expressions discussed in the previous section, this type of ODE is an **inhomogeneous** equation.

To show the approach in solving an ODE with an additional term, we will discuss a simplified linear model for open and closed ion channels (see Fig. 1.1); in this case there is a finite amount of ion channels available and we use O and C as fractions of open and closed gates:

INSET 2 EXACTLY HERE

Let us assume that, as in the famous Hodgkin and Huxley (1952) formalism, the tendency for gates to open or close depends on the membrane potential. Thus at a given membrane potential E, the rate constants governing the opening and closing dynamics are m_E, h_E. Here, we simplify notation by omitting the subscripts E. The dynamics for the fraction of open gates at the given membrane potential is governed by:

$$dO/dt = mC - hO. \qquad (9.15)$$

Since O and C are fractions of the total number of available gates, we have $C = 1 - O$, this allows us to write Equation (9.15) as $dO/dt = m(1 - O) - hO$, which can be simplified to:

$$dO/dt = \dot{O} = m - (m + h)O. \qquad (9.16)$$

In this case our 'forcing term' is not really originating from an external force, it is a constant (m) originating from the $1 - O$ term. The general approach in finding the solution in the case of a forcing term is to solve the equation without the forcing term first; this will give the intrinsic, unforced solution of the dynamics. Then one solves the full equation, usually by assuming that there is a **particular** solution in the same form as the forcing term. Because we assumed m being a constant in this example, we also assume that there is a solution in the form of a constant. After we found both solutions for the unforced and forces case, we apply again the rule that any scaled superposition of solutions is a solution of a linear system (***CH 13***). Following this recipe, we combine the solution of the unforced ODE with the particular solution and obtain the **general solution** of the forced ODE. This solution will again contain constants that must be found from boundary values. Applying this approach, we know that the solution of the unforced ODE is in the same form as Equation (9.8), i.e.

$$O(t) = Ae^{-(m+h)t}. \qquad (9.17)$$

Since the forcing term in this example is a constant, we assume that the particular solution O_P is also a constant. Since the derivative of a constant vanishes, we know that $dO_P/dt = 0$. Plugging that into equation (9.16), we find that:

$$O_P(t) = \frac{m}{m+h}, \text{ or a scaled version of this result.} \qquad (9.18)$$

The combination of Equations (9.17) and (9.18) leads to our general solution O_G:

$$O_G(t) = Ae^{-(m+h)t} + B\frac{m}{m+h}. \qquad (9.19)$$

Now let us assume that at $t = 0$ all channels are closed, i.e. $O_G(0) = 0$. Further, let us assume that there is equilibrium at $t = \infty$, i.e. $\dot{O}_G = 0$ (note the dot!) at $t = \infty$. The latter condition means that the exponential term at $t = \infty$ vanishes (since m and h are both positive rate constants), and we can determine from Equation (9.16) that the equilibrium state corresponds to $\frac{m}{m+h}$, i.e. we find that $B = 1$. Going back to the initial state at $t = 0$, we have the exponential term $e^{-(m+h)t} = 1$, thus we find that $A = -\frac{m}{m+h}$. Plugging the results for A and B into Equation (9.19) gives:

$$O_G(t) = \frac{m}{m+h}\left(1 - e^{-(m+h)t}\right). \qquad (9.20)$$

Now let's investigate a second order case. For a second order ODE with a forcing term, we follow exactly the same procedure as we did above for the first order one. Since we established that the potential dynamics include oscillation, let's apply an oscillatory input to the dynamics in Equation (9.10) with amplitude α and radial frequency ω_F, so that we get:

$$\ddot{S} + b\dot{S} + cS = \alpha\cos\omega_F t. \qquad (9.21)$$

We already determined the solution for the homogeneous equation (Equation (9.13)). Now, as we did for the 1st order ODE, we assume there is a particular solution in the form of the forcing term. To make this straightforward, as you will notice below, we will replace the forcing term $\alpha\cos\omega_F t$ by the more general expression for an oscillation $\alpha e^{j\omega_F t}$, and we then assume that the particular solution is in the form $de^{j\omega_F t}$, where d is a constant. Thus now we have:

$$\ddot{S} + b\dot{S} + cS = \alpha e^{j\omega_F t}. \qquad (9.22)$$

In this version, S is a complex variable. Strictly speaking it is a bit sloppy to use S for both the complex variable and its real part, but we will leave it so since there is no serious ambiguity in the following.

NOTE: We use here the imaginary exponential term $e^{j\omega_F t}$, representing a combination of sine and cosine waves - recall Euler's formula: $e^{j\omega_F t} = \cos\omega_F t + j sin\omega_F t$. Alternatively, we could also employ $a\cos\omega_F t + b sin\omega_F t$ for the particular solution and determine a and b by following the same procedure as below.

Differentiating the complex exponential expression for the particular solution with respect to time, we find $j\omega_F de^{j\omega_F t}$ and $-\omega_F^2 de^{j\omega_F t}$ for the 1st and 2nd derivatives. We now plug this into the expression left of the equal sign of Equation (9.22), and we get:

$$\ddot{S} + b\dot{S} + cS = -\omega_F^2 de^{j\omega_F t} + j\omega_F bde^{j\omega_F t} + cde^{j\omega_F t} = [-\omega_F^2 + j\omega_F b + c]de^{j\omega_F t}$$

Since this must be equal to the right hand side of the expression in Equation (9.22), we now have determined that $[-\omega_F^2 + j\omega_F b + c]d = \alpha$, which enables us to express how unknown constant d depends on the parameters of the ODE:

$$d = \frac{\alpha}{[-\omega_F^2 + j\omega_F b + c]}.$$

Obviously, d is a complex number. Thus the complex solution for Equation (9.22) is:

$$S = \frac{\alpha}{[-\omega_F^2 + j\omega_F b + c]} e^{j\omega_F t}. \quad (9.23)$$

Recall that the cosine expression is the real component of the Euler formula. Thus, since the forcing term in Equation (9.21) is a cosine, we only need the real part of the solution in the above expression. The real part can be obtained by writing the constant factor and the exponential in Equation (9.23) in their real and imaginary components:

$$\mathrm{S} = \left[\frac{\alpha(c-\omega^2)}{(c-\omega^2)^2 + b^2\omega^2} - j\frac{\alpha b\omega}{(c-\omega^2)^2 + b^2\omega^2}\right][\cos\omega_F t + j\sin\omega_F t].$$

Now it is straightforward to determine the real part of S, i.e. the particular solution of Equation (9.21):

$$\mathrm{S} = \frac{\alpha(c-\omega^2)}{(c-\omega^2)^2 + b^2\omega^2}\cos\omega_F t + \frac{\alpha b\omega}{(c-\omega^2)^2 + b^2\omega^2}\sin\omega_F t. \quad (9.24)$$

Assuming we found a solution for the homogeneous equation as in Equation (9.13a), our general solution becomes:

$$S = Ae^{k_1 t} + Be^{k_2 t} + \frac{\alpha(c-\omega^2)}{(c-\omega^2)^2 + b^2\omega^2}\cos\omega_F t + \frac{\alpha b\omega}{(c-\omega^2)^2 + b^2\omega^2}\sin\omega_F t. \quad (9.25)$$

The algebra may seem a bit cumbersome, but it is all straightforward algebra. You can check this, when using numbers for the parameters.

9.5 Representation of Higher Order ODEs as a Set of 1st Order Ones

The highest order ODE we discussed thus far is the second order one such as in Equation (9.10). Although one always makes a serious attempt to keep things as simple as possible, if we deal with real dynamical systems, higher order ODEs are not exceptional. One extremely useful technique to solve these cases is that an n^{th} order system can be decomposed into a set of n 1st order ODEs. Here we repeat the 2nd order ODE of Equation (9.10) for convenience:

$$\ddot{S} + b\dot{S} + cS = 0.$$

If we now define $S_1 = S$ and $S_2 = \dot{S} = \dot{S}_1$, we have $\dot{S}_2 = \ddot{S}_1$. Now we can rewrite the 2nd order ODE as two 1st order differential equations:

$$\begin{aligned} \dot{S}_1 &= S_2 \\ \dot{S}_2 &= -cS_1 - bS_2, \end{aligned} \quad (9.26)$$

or if we use the boldface notation $\boldsymbol{S}$ to denote the vector $[S_1\, S_2]^T$, where T indicates the transpose of the row vector, we can simplify the two expressions into a single vector expression:

$$\dot{\boldsymbol{S}} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \boldsymbol{S}, \text{ or for the general case,} \tag{9.27}$$
$$\dot{\boldsymbol{S}} = A\boldsymbol{S}, \text{ using } A \text{ to denote a matrix.}$$

Even for a 2nd order ODE this change of notation is attractive because we can now use straightforward linear algebra to obtain the solution. Analogous to what we did for solving the ODEs above, we assume there is a solution in the form

$$e^{\lambda t}\boldsymbol{V}. \tag{9.28}$$

In this expression λ is the rate of change and note that $\boldsymbol{V}$ is a vector. If we differentiate the expression for the solution, we get $\lambda e^{\lambda t}\boldsymbol{V}$. Now we substitute the assumed solution and its derivative in the generic form of Equation (9.27):

$$\lambda e^{\lambda t}\boldsymbol{V} = e^{\lambda t}A\boldsymbol{V}.$$

Dividing by the (nonzero) exponential gives:

$$A\boldsymbol{V} = \lambda\boldsymbol{V}. \tag{9.29}$$

As you may recall, this is a well know equation in linear algebra where $\boldsymbol{V}$ is an eigenvector of matrix A and λ the associated eigenvalue. Following the standard procedure of linear algebra, the eigenvalues corresponding to Equation (9.29) can be found by solving the **characteristic equation**:

$$|A - \lambda\boldsymbol{I}| = 0, \tag{9.30}$$

with $|\ldots|$ denoting the determinant and $\boldsymbol{I}$ the identity matrix (in this example a 2×2 matrix).

Let's consider a few examples.
We start using the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.
First let us determine the solutions for which $\dot{S_1}$ and $\dot{S_2}$ are zero. This gives the solutions $S_2 = -S_1/3$ and $S_2 = -S_1$, the so-called null-clines for S_1 and S_2 respectively.
Next, using Equation (9.30), the eigenvalues can be found from $\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$. If we expand this expression and factorize the result, we have $(\lambda - 4)(\lambda + 1) = 0$. This gives us the two solutions for the eigenvalues: 4 and -1. Finding the eigenvectors associated with these eigenvalues produces eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ respectively (Fig. 9.1A). Thus the solution associated with the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ grows since it is governed by a positive exponential (i.e. a

positive eigenvalue, $\lambda = 4$) and the other one associated with $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ decays because the eigenvalue is negative ($\lambda = -1$).

We have used the example of the 2nd order ODE: $\ddot{S} + b\dot{S} + cS = 0$ (Eq. (9.10)) and rewrote this expression as $\dot{\boldsymbol{S}} = A\boldsymbol{S}$ (Eq. (9.27)). Recall that matrix $A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$, and the solutions $k_{1,2}$ were found from the characteristic equation (see also Eqs. (9.11), (9.12)): $k_{1,2} = \left(-b \pm \sqrt{b^2 - 4c}\right)/2$. Now note that the trace T and determinant D of matrix A are $-b$ and c respectively. This allows us to write the expressions for the solutions for $\lambda_{1,2}$ as:

$$\lambda_{1,2} = \left(T \pm \sqrt{T^2 - 4D}\right)/2. \quad (9.31)$$

We can use Eq (9.31) to show that trace and determinant for matrices A and $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ are the same, i.e.: $T = \lambda_1 + \lambda_2$; $D = \lambda_1 \lambda_2$. Thus we have two solutions for the characteristic equation, i.e. the **eigenvalues** k_1 and k_2. Since we have a linear differential equation, the solution to the ODE is a linear combination determined by the two eigenvalues (see Eq. (9.13a)). Now that we changed notation for the 2nd order ODE, we can also change notation for the scenarios listed in Eq. (9.12). If we now consider Eq. (9.31) and analyze the effects of T and D in addition to the value of the term under the square root, $T^2 - 4D$, we can further subdivide these scenarios and we get the following possibilities:

(1) two real solutions for λ if $(T^2 - 4D) > 0$, (9.32)
   (a) if $D < 0$, one solution is positive and one is negative,
   (b) if $D > 0$, both solutions are either positive or negative,
      (i) if $T < 0$ both solutions are negative,
      (ii) if $T > 0$ both solutions are positive;
(2) two complex conjugate solutions for λ if $(T^2 - 4D) < 0$,
   (a) if $T < 0$, the real part of the solution is negative,
   (b) if $T > 0$, the real part of the solution is positive,
   (c) if $T = 0$, the solutions are imaginary;
(3) just one real solution for λ if $(T^2 - 4D) = 0$,
   (a) with two independent eigenvectors,
   (b) with one eigenvector.

In the following we show how you can examine these alternative possibilities in MATLAB. In this example we take the generic notation of a 2nd order ODE: $\dot{\boldsymbol{S}} = A\boldsymbol{S}$ (Eq. (9.27)). Next, we use S1 and S2 as the components of the vector $\boldsymbol{S}$, and dS1 and dS2 for the components of its derivative $\dot{\boldsymbol{S}}$. In the MATLAB command sequence below, the command meshgrid defines the range and precision with which we plot the result. The coefficients used to compute dS1 and dS2 below are example entries of matrix A, here we use $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. The quiver command plots the values of derivatives dS1 and dS2 as a vector field in the S1-S2 plane - recall that this is the phase plane since $S_1 = S$ and $S_2 = \dot{S}$.

To obtain the plot in Fig. 9.1A type in the following at the MATLAB command prompt (or run the script pr9_1.m).

```
[S1,S2] = meshgrid(-3:.5:3,-3:.5:3);
dS1=1*S1+3*S2;
dS2=2*S1+2*S2;
figure;
quiver(S1,S2,dS1,dS2,'k');
axis([-3 3 -3 3])
axis('square')
```

If we categorize this example using its matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, compute the values for $T = 3$, $D = -4$ and $T^2 - 4D = 25$, using Equation (9.32), we note that this represents **case 1a**: two positive real solutions for the characteristic equation, one positive and one negative. This is easily confirmed in the result depicted in Fig. 10A, we have a so-called saddle point with one stable manifold associated with a negative solution and one unstable one associated by the positive solution. Using the same MATLAB procedure, it is easy to explore some other values for A, for example try $A = \begin{bmatrix} 2 & -4 \\ 2 & 3 \end{bmatrix}$ ***by replacing the lines for dS1 and dS2 in the above sequence of commands by the following ones:***

```
dS1=2*S1-4*S2;
dS2=2*S1+3*S2;
```

This will generate a result as depicted in Fig. 9.1B. Interestingly, in this example the vector arrangement in the S1-S2 plane follows a different pattern, i.e. spiral trajectories moving away from the origin. If we categorize this example using its matrix $A = \begin{bmatrix} 2 & -4 \\ 2 & 3 \end{bmatrix}$, compute the values for $T = 5$, $D = 14$ and $T^2 - 4D = -31$, and using Equation (9.32), we note that this represents **case 2b**: two complex conjugate solutions with positive real values. When we relate this to the phase space plot in Fig 9.1B, the complex solutions govern the oscillatory behavior and the positive real values cause the oscillations to grow, away from the fixed point, a so-called unstable spiral or also called spiral source.

Fig. 9.1

The two results we obtained above and depicted in Figure 9.1A, B are two examples out of the possible behaviors generated by a 2^{nd} order ODE, i.e. the dynamics one may observe in a 2D phase plane. Depending on the values we pick for matrix *A*, we can get any of the behaviors we summarized in Equation (9.32). These examples show that the values for trace (*T*) and determinant (*D*) can be used to diagnose the dynamical behavior of the ODE. Figure 9.1C depicts the scenarios listed in Equation (9.32) in a diagram of *T* and *D*. Each of the numbers in Figure 9.1C corresponds to the numbering in Equation (9.32).

Using the same numbers for the different scenarios as we used in Equation (9.32) and Figure 9.1C, depending on *T* and *D*, we find the following dynamics:

- (1a) a saddle point,
- (1bi) a stable node,
- (1bii) an unstable node,
- (2a) a stable spiral, also called spiral sink,

(2b) an unstable spiral, also called spiral source,
(2c) a borderline case, the center
(3a) borderline cases, a star node if $\lambda \neq 0$, or no dynamics if $\lambda = 0$
(3b) borderline case, a degenerate node.

You can create a phase plane plot of each case by changing variables dS1 and dS2 in the above MATLAB example. For instance, a star node (case 3a) can be observed when using:

```
dS1=1*S1+0*S2;
dS2=0*S1+1*S2;
```

Note that this case is located on the horizontally oriented parabola in Figure 9.1C: since *T*=2 and *D*=1, thus the condition $T^2 - 4D = 0$ is satisfied.

A degenerate node (case 3b) can be observed when using:

```
dS1=1*S1+1*S2;
dS2=0*S1+1*S2;
```

Note that also in this case *T*=2 and *D*=1, however, there is an additional non-zero off-diagonal element in matrix *A* now.

A center (case 2c) can be observed when using:

```
dS1=0*S1+1*S2;
dS2=-1*S1+0*S2;
```

Note that in this case T=0 and D=1, satisfying the condition for the center depicted in Fig. 9.1.

9.6 Transforms to Solve ODEs

Another, often convenient, way to solve ODEs is to use transforms: the Fourier, Laplace, or *z* transform. The idea here is to transform the ODE from the time or spatial domain into another domain and obtain the solution in the transformed domain. Next, the solution is inverse transformed into the original (time or spatial) domain, and voilà there we have the solution (see Fig. 12.1). For details and several examples of this approach see ***CH 12*** and ***CH 16***.

$$A + B \underset{k_2}{\overset{k_1}{\rightleftarrows}} C$$

INSET 1

$$C \underset{h}{\overset{m}{\rightleftarrows}} O$$

INSET 2

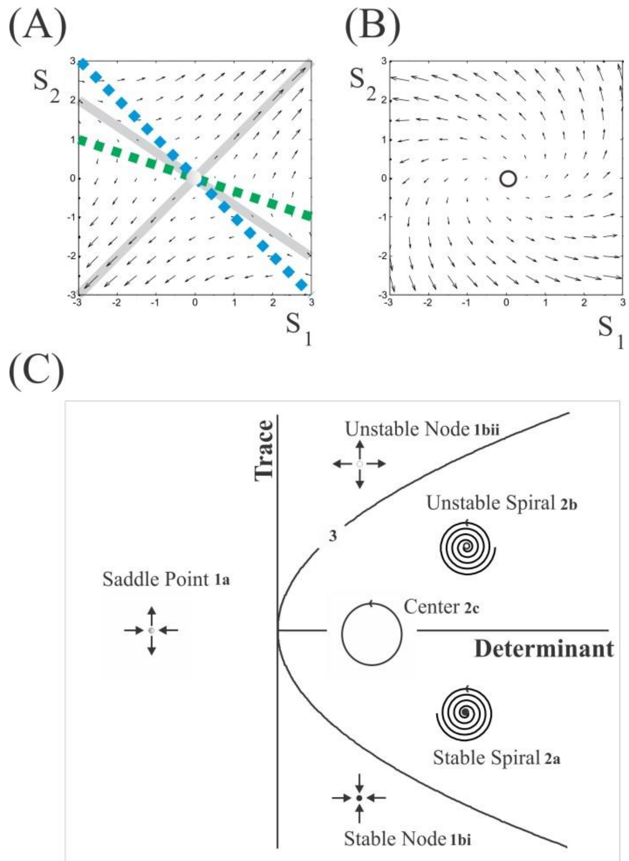

Fig. 9.1
(A), (B) Examples of phase plane plots for the 2nd order ODE: $\ddot{S} + b\dot{S} + cS = 0$. Note that in both plots S is defined as S1 and $\dot{S}$ as S2. In panel A, we show a saddle point with one stable and one unstable manifold (the grey lines). The colored lines are the null clines for S1 (green) and S2 (blue). This case represents scenario 1a in Equation (32). In panel B we show a scenario with unstable (growing) oscillatory behavior. This case represents scenario 2b in Equation (32). (C) Diagram of the type and stability of the dynamics for the 2nd order ODE in Equation (9.27) as a function of the values for the trace T (ordinate) and determinant D (abscissa) of matrix A. For negative values of the determinant we have saddle points. For a positive determinant, the values for the trace and $T^2 - 4D$ determines the solution's stability. The horizontally oriented parabola (governed by $T^2 - 4D$) separates nodes from spirals and centers. The labels 1a – 3 correspond to the scenarios summarized in Eq. (9.32). On this parabola (label 3), the eigenvalues are identical in all directions giving rise to so-called star (independent pair of eigenvectors), or degenerate (only one eigenvector) nodes.